CATEGORICAL STRUCTURES IN NON-COMMUTATIVE MEASURE THEORY

(*Dedicated to Gottfried T. Ruttimann, in memoriam*)

Pedro Morales and Francisco García Mazarío

Département de Mathématiques et d'Informatique
Université de Sherbrooke
Sherbrooke, Québec, J1K 2R1, Canada

Departamento de Matemática Aplicada (E.U. Informática)
Universidad Politécnica de Madrid
Carretera de Valencia, Km. 7, 28031, Madrid, Spain
e-mail: gmazario@eui.upm.es

Abstract

In this paper, we investigate effect algebras and base normed spaces from the categorical point of view. We prove that the category of effect algebras is complete and cocomplete as well as the category of base normed spaces is complete, and discuss the contravariant functor from the category of effect algebras to the category of base normed spaces.

1. Introduction

The use and development of positive operator valued measures in stochastic quantum mechanics (see [5]) gives impetus to a new class of orthostructures now called *effect algebras*. We are interested in these partial algebras because they carry, in a natural way, orthogonally additive measures and allow us to introduce probability measures on an effect algebra.

The purpose of this paper is to consider effect algebras and base normed spaces as categories by choosing suitable morphisms and to investigate the basic properties of these categories.

The paper is organized as follows:

In Section 2, we present some basic notions on Category Theory following mainly references [1, 4, 14-16] and we refer to [2, 19] for additional information. Section 3 introduces the category $\mathcal{A}$ of effect algebras and establishes that $\mathcal{A}$ have infinite direct sums and infinite direct products and two morphisms in $\mathcal{A}$ have a coequalizer and an equalizer. In Section 4, we give some algebraic notions in order to define a base normed space, we present two important examples, we introduce the category $\mathcal{B}$ of base normed spaces and we show that $\mathcal{B}$ has infinite direct products and two morphisms in $\mathcal{B}$ have an equalizer. Section 5, after defining the notions of probability measure and Jordan measure on an effect algebra, describes the contravariant functor T from $\mathcal{A}$ to $\mathcal{B}$ and establishes the continuity of T as the main result of the paper.

2. Basic Notions on Category Theory

The framework of our presentation of category theory is the standard Gödel-Bernays axioms for set theory (see [11] or [16, Appendix]).

Consider a system $\mathcal{C}$ consisting of a class $\mathrm{Ob}\,\mathcal{C}$, together with a class $\mathrm{Mor}\,\mathcal{C}$ which is a disjoint union of the form

$$\mathrm{Mor}\,\mathcal{C} = \bigcup \{[A, B] : A, B \in \mathrm{Ob}\,\mathcal{C}\}.$$

We say that $\mathcal{C}$ is a category if the following axioms are satisfied:

(i) Each $[A, B]$ is a set (possibly empty).

(ii) For each triple (A, B, C) of elements of $\mathrm{Ob}\,\mathcal{C}$ there is a map

$$\theta(A, B, C) : [A, B] \times [B, C] \to [A, C]$$

called the *composition map* (if $f \in [A, B]$, and $g \in [B, C]$, we shall denote by $g \circ f$ the image $\theta(A, B, C)(f, g)$ of the pair (f, g) under $\theta(A, B, C)$).

(iii) The composition map is associative, that is, if $f \in [A, B]$, $g \in [B, C]$ and $h \in [C, D]$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

(iv) The composition map possesses identities, that is, for every $A \in \mathrm{Ob}\,\mathcal{C}$ there is an element $1_A \in [A, A]$ such that for every $B \in \mathrm{Ob}\,\mathcal{C}$, every $f \in [A, B]$ and every $g \in [B, A]$ we have $f \circ 1_A = f$ and $1_A \circ g = g$.

The elements of $\mathrm{Ob}\,\mathcal{C}$ are called *objects* of $\mathcal{C}$ and the elements of $\mathrm{Mor}\,\mathcal{C}$ are called *morphisms* in $\mathcal{C}$. If $f \in [A, B]$ we call A the domain of f and B the codomain, and we say that f is a morphism from A to B and we write $f : A \to B$ or $A \xrightarrow{f} B$. We note that 1_A is the only identity for A (see [15, p. 2]). If $\mathrm{Ob}\,\mathcal{C}$ is a set, then the category $\mathcal{C}$ will be called small, and in this case $\mathrm{Mor}\,\mathcal{C}$ is also a set.

Three important examples of categories are the following:

(1) The category $\mathcal{S}$ whose class of objects is the class of all sets, where $[A, B]$ is the set of all maps from A to B with the agreement that, for every set X, there is a unique map from $\varnothing$ to X, is called the category of sets.

(2) Let J be any non-empty set. The category $\mathrm{Disc}\,J$ whose class of objects is the set J, where $[j, k]$ is the set consisting of a single element 1_j if $j = k$ and is the empty set otherwise, is called the discrete category determined by J.

(3) Let $(I, \leq)$ be a preordered set. The category Ord I whose class of objects is the set I, where $[i, j]$ is empty unless $i \leq j$ in which case $[i, j]$ is the set consisting of a single element, is called the preordered category determined by $(I, \leq)$.

When $\mathcal{C}$ is a category and there is danger of confusion we shall write $[A, B]_{\mathcal{C}}$ in place of $[A, B]$.

A category $\mathcal{C}'$ is called a *subcategory* of a category $\mathcal{C}$ provided:

(i) $\operatorname{Ob} \mathcal{C}'$ is a subclass of $\operatorname{Ob} \mathcal{C}$.

(ii) $[A, B]_{\mathcal{C}'} \subseteq [A, B]_{\mathcal{C}}$ for all $A, B \in \operatorname{Ob} \mathcal{C}'$.

(iii) The composition of any two morphisms in $\mathcal{C}'$ is the same as their composition in $\mathcal{C}$.

(iv) 1_A is the same in $\mathcal{C}'$ as in $\mathcal{C}$ for all $A \in \operatorname{Ob} \mathcal{C}'$.

If, furthermore, $[A, B]_{\mathcal{C}'} = [A, B]_{\mathcal{C}}$ for all $A, B \in \operatorname{Ob} \mathcal{C}'$, then we say that $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$.

The *dual category* of a category $\mathcal{C}$, denoted $\mathcal{C}^*$, is the category whose class of objects is $\operatorname{Ob} \mathcal{C}$, where $[A, B]_{\mathcal{C}^*} = [B, A]_{\mathcal{C}}$ and, for $f \in [A, B]_{\mathcal{C}^*}$ and $g \in [B, C]_{\mathcal{C}^*}$ the composition $g \circ f$ in $\mathcal{C}^*$ is defined as the composition $f \circ g$ in $\mathcal{C}$.

Let $\mathcal{C}$ be a category, let $A, B \in \operatorname{Ob} \mathcal{C}$ and let $f : A \to B$. Then we say that

(a) f is a *monomorphism* if for any two morphisms $g_1, g_2 : C \to A$ the equality $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.

(b) f is an *epimorphism* if for any two morphisms $h_1, h_2 : B \to C$ the equality $h_1 \circ f = h_2 \circ f$ implies $h_1 = h_2$.

(c) f is an *isomorphism* if there is a morphism $g : B \to A$ with $g \circ f = 1_A$ and $f \circ g = 1_B$.

(If such a morphism exists, then it is unique by [1, Proposition 3.10]). We say also that the objects A and B are isomorphic and we write $A \cong B$.

Let $\mathcal{C}$ be a category and let $f_1, f_2 : A \to B$ be two morphisms in $\mathcal{C}$. We say that

(a) The pair (K, k) consisting of an object K of $\mathcal{C}$ and a morphism $k : K \to A$ in $\mathcal{C}$ is called an *equalizer* for f_1 and f_2 if the following conditions are fulfilled:

(i) $f_1 \circ k = f_2 \circ k$.

(ii) Whenever $f : K' \to A$ is a morphism in $\mathcal{C}$ such that $f_1 \circ f = f_2 \circ f$, there exists a unique morphism $f' : K' \to K$ with $k \circ f' = f$.

(b) The pair (H, h) consisting of an object H of $\mathcal{C}$ and a morphism $h : B \to H$ in $\mathcal{C}$ is called a *coequalizer* for f_1 and f_2 if the following conditions are fulfilled:

(i') $h \circ f_1 = h \circ f_2$.

(ii') Whenever $g : B \to H'$ is a morphism in $\mathcal{C}$ such that $g \circ f_1 = g \circ f_2$, there exists a unique morphism $g' : H \to H'$ with $g' \circ h = g$.

It is easy to show that the pairs (K, k) and (H, h) are unique up to isomorphism, k is a monomorphism and h is an epimorphism (see [1, Proposition 7.53] and [4, Proposition 2.13] noting that equalizer and coequalizer are dual notions).

An object A of a category $\mathcal{C}$ is called initial if the set $[A, X]$ consists of a single element for any object X of $\mathcal{C}$. It is easy to see that any two initial objects of $\mathcal{C}$ are isomorphic (see [16, Lemma 1, p. 22]). For example, in the category $\mathcal{S}$ the empty set is an initial object.

An object B of a category $\mathcal{C}$ is called *terminal* if the set $[Y, B]$ consists of a single element for any object Y of $\mathcal{C}$. By duality, any two terminal objects of $\mathcal{C}$ are isomorphic. For example, in the category $\mathcal{S}$ any singleton is a terminal object.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant* (resp. *contravariant*) functor from $\mathcal{C}$ to $\mathcal{D}$, denoted $T : \mathcal{C} \to \mathcal{D}$, is an assignment of an object $T(A)$ of $\mathcal{D}$ to each object A of $\mathcal{C}$ and a morphism $T(f) \in [T(A), T(B)]_{\mathcal{D}}$ (resp. $T(f) \in [T(B), T(A)]_{\mathcal{D}}$) to each morphism $f \in [A, B]_{\mathcal{C}}$ subject to the following conditions:

(a) For every $A \in \operatorname{Ob} \mathcal{C}$, we have $T(1_A) = 1_{T(A)}$.

(b) If $f \in [A, B]_{\mathcal{C}}$ and $g \in [B, C]_{\mathcal{C}}$, then $T(g \circ f) = T(g) \circ T(f)$ (resp. $T(g \circ f) = T(f) \circ T(g)$).

Clearly, two functors may be composed in an obvious way and the composition of two contravariant functors is a covariant functor. The contravariant functor $S : \mathcal{C}^* \to \mathcal{C}$ defined by $S(A) = A$ for every object A of $\mathcal{C}^*$ and $S(f) : B \to A$ for every morphism $f : A \to B$ in $\mathcal{C}^*$, is called the *dualizing functor* of $\mathcal{C}$. Then to each contravariant functor $T : \mathcal{C} \to \mathcal{D}$ we can associate the covariant functor $T \circ S : \mathcal{C}^* \to \mathcal{D}$, and vice-versa, so that the general study of contravariant functors can be reduced to the study of covariant functors.

A covariant functor $T : \mathcal{C} \to \mathcal{D}$ is called *constant* if there exists an object D of $\mathcal{D}$ such that $T(A) = D$ for all $A \in \operatorname{Ob} \mathcal{C}$ and for each morphism $f : A \to B$ in $\mathcal{C}$ we have $T(f) = 1_D$. Sometimes we say that T is a *constant functor of value D*.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let S and T be covariant (resp. contravariant) functor from $\mathcal{C}$ to $\mathcal{D}$. A *natural transformation* η from S to T, denoted by $\eta : S \to T$, is a prescription which assigns to each object A of $\mathcal{C}$

a morphism $\eta_A : S(A) \to T(A)$ in $\mathcal{D}$ such that for every morphism $f : A \to B$ in $\mathcal{C}$ the following diagram:

$$\begin{tikzcd} S(A) \arrow[r, "\eta_A"] \arrow[d, "S(f)"'] & T(A) \arrow[d, "T(f)"] \\ S(B) \arrow[r, "\eta_B"] & T(B) \end{tikzcd} \quad \left(\text{resp. } \begin{tikzcd} S(A) \arrow[r, "\eta_A"] & T(A) \\ S(B) \arrow[r, "\eta_B"] \arrow[u, "S(f)"] & T(B) \arrow[u, "T(f)"'] \end{tikzcd} \right)$$

commutes. When this holds, we also say that η is natural in A. If η_A is an isomorphism for every $A \in \operatorname{Ob} \mathcal{C}$, then η is called a *natural equivalence* and we write $\eta : S \cong T$.

If S, T and U are three covariant functors from $\mathcal{C}$ to $\mathcal{D}$ and if $\eta : S \to T$ and $\rho : T \to U$ are two natural transformations, then we have a composition $\rho \circ \eta : S \to U$ defined by the rule $(\rho \circ \eta)_A = \rho_A \circ \eta_A$ for all $A \in \operatorname{Ob} \mathcal{C}$.

Suppose now that $\mathcal{C}$ is a small category. Then the category $[\mathcal{C}, \mathcal{D}]$ whose class of objects is the class of all covariant functors from $\mathcal{C}$ to $\mathcal{D}$, where $[S, T]$ is the set (according to [15, p. 63]) of all natural transformations from S to T, is called the *category* of functors from $\mathcal{C}$ to $\mathcal{D}$. In the special case where $\mathcal{C} = \operatorname{Disc} J$, it is easy to see that every object T of $[\mathcal{C}, \mathcal{D}]$ is a family $(T(j))_{j \in J}$ of objects of $\mathcal{D}$ and every natural transformation $\eta \in [S, T]$ is a family $(\eta_j : S(j) \to T(j))_{j \in J}$ of morphisms in $\mathcal{D}$ (see [14, pp. 69-70]).

Let I be a directed set and let $\mathcal{C} = \operatorname{Ord} I$. Then it is easy to see that every object T of $[\mathcal{C}, \mathcal{D}]$ is of the form $T = (T(i), f_{ij})_I$, where each $T(i)$ is an object of $\mathcal{D}$ and each f_{ij} is a morphism from $T(i)$ to $T(j)$ for all $i, j \in I$ with $i \leq j$, such that $f_{ii} = 1_{T(i)}$ and $f_{jk} \circ f_{ij} = f_{ik}$ for $i \leq j \leq k$ in I. Also, every object S of the category $[\mathcal{C}^*, \mathcal{D}]$ is of the form $S = (S(i), g_{ij})_I$ where each $S(i)$ is an object of $\mathcal{D}$ and each g_{ij} is a morphism from $S(j)$ to

$S(i)$ for all $i, j \in I$ with $i \leq j$, such that $g_{ii} = 1_{S(i)}$ and $g_{ij} \circ g_{jk} = g_{ik}$ for $i \leq j \leq k$.

Now let $\mathcal{C}$ be a category, let $\mathcal{D}$ be a small category and let $\Delta : \mathcal{C} \to [\mathcal{D}, \mathcal{C}]$ be a covariant functor. We say that Δ is a diagonal functor if it satisfies the following conditions:

(i) For each object A of $\mathcal{C}$, the covariant functor $\Delta(A) : \mathcal{D} \to \mathcal{C}$ is a constant functor of value A.

(ii) For each morphism $f : A \to B$ in $\mathcal{C}$, the natural transformation $\Delta(f) \in [\Delta(A), \Delta(B)]$ has the same value f at each object D of $\mathcal{D}$.

Let S be a covariant functor from a category $\mathcal{D}$ to a category $\mathcal{C}$ and let C be an object of $\mathcal{C}$. We define

(a) The *comma category* $(C \downarrow S)$ of objects S-under C as the category whose objects are pairs (D, f) with $D \in \operatorname{Ob} \mathcal{D}$ and $f : C \to S(D)$ and whose morphisms from (D, f) to (D', f') are those morphisms $h : D \to D'$ in $\mathcal{D}$ for which $f' = S(h) \circ f$.

(b) The *comma category* $(S \downarrow C)$ of objects S-over C in a similar way (for the general construction see [1, Exercise 3K]).

(c) A *universal morphism from* C to S as a pair (U, u) consisting of an object U of $\mathcal{D}$ and a morphism $u : C \to S(U)$ in $\mathcal{C}$ such that to every pair (U', u') with U' an object of $\mathcal{D}$ and $u' : C \to S(U')$ a morphism in $\mathcal{C}$, there is a unique morphism $f : U \to U'$ in $\mathcal{D}$ with $S(f) \circ u = u'$.

(d) A *universal morphism from S to C* as a pair (V, v) consisting of an object V of $\mathcal{D}$ and a morphism $v : S(V) \to C$ in $\mathcal{C}$ such that to every pair (V', v') with V' an object of $\mathcal{D}$ and $v' : S(V') \to C$ a morphism in $\mathcal{C}$, there is a unique morphism $f : V' \to V$ in $\mathcal{D}$ with $v \circ S(f) = v'$.

It is easy to see that a universal morphism (U, u) from C to S is an initial object of the comma category $(C \downarrow S)$, and therefore (U, u) is unique up to isomorphism. Moreover, a universal morphism (V, u) from S to C is a terminal object of the comma category $(S \downarrow C)$, and therefore (V, v) is unique up to isomorphism.

Let J be a non-empty set, let $\mathcal{D} = \text{Disc } J$, let $\mathcal{C}$ be a category, let $\Delta : \mathcal{C} \to [\mathcal{D}, \mathcal{C}]$ be the corresponding diagonal functor and let $(C_j)_{j \in J}$ be a family of objects of $\mathcal{C}$. Then $C = (C_j)_{j \in J}$ is an object of the functor category $[\mathcal{D}, \mathcal{C}]$. We say that

(a) A universal morphism (U, u) from C to Δ is called a *direct sum* of the family $C = (C_j)_{j \in J}$. Since u is a natural transformation from C to $\Delta(U)$, we can write $u = (u_j : C_j \to U)_{j \in J}$ where each u_j is a morphism in $\mathcal{C}$. They are called *canonical injections* despite the fact that they are not in general monomorphisms (see [4, Example, pp. 38-39]). Sometimes we write $U = \coprod_{j \in J} C_j$ and we call the object U of $\mathcal{C}$ a direct sum of the family $(C_j)_{j \in J}$.

(b) The category $\mathcal{C}$ has *infinite direct sums* if every family of objects of $\mathcal{C}$ possesses at least one direct sum.

(c) A universal morphism (V, v) from Δ to C is called a *direct product* of the family $(C_j)_{j \in J}$. Since v is a natural transformation from $\Delta(V)$ to C, we can write $v = (v_j : V \to C_j)_{j \in J}$ where each v_j is a morphism in $\mathcal{C}$. They are called *canonical projections* despite the fact that they are not in general epimorphisms (see [1, Example 10.20(3)] and [7, Exercise 17(c), p. 160]). Sometimes we write $V = \prod_{j \in J} C_j$ and we call the object V of $\mathcal{C}$ a direct product of the family $(C_j)_{j \in J}$.

(d) The category $\mathcal{C}$ has *infinite direct products* if every family of objects of $\mathcal{C}$ possesses at least one direct product.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories such that $\mathcal{D}$ is small, let $\Delta : \mathcal{C} \to [\mathcal{D}, \mathcal{C}]$ be the corresponding diagonal functor and let T be a covariant functor from $\mathcal{D}$ to $\mathcal{C}$. Then T is an object of the functor category $[\mathcal{D}, \mathcal{C}]$. We say that

(a) A universal morphism (M, λ) from Δ to T is called a *limit of the functor* T. Since λ is a natural transformation from $\Delta(M)$ to T, it follows from the definition that $\lambda = (\lambda_A : M \to T(A))_{A \in \mathrm{Ob}\,\mathcal{D}}$ and from the naturality of λ in A, it follows that $\lambda_B = T(f) \circ \lambda_A$ for each morphism $f : A \to B$ in $\mathcal{D}$. Sometimes we write $M = \varprojlim T$ and we call the object M of $\mathcal{C}$ a limit of the functor T.

(b) The category $\mathcal{C}$ is *complete* if every covariant functor from a small category to $\mathcal{C}$ has a limit.

(c) A universal morphism (L, τ) from T to Δ is called a *colimit of the functor* T. Since τ is a natural transformation from T to $\Delta(L)$, it follows from the definition that $\tau = (\tau_A : T(A) \to L)_{A \in \mathrm{Ob}\,\mathcal{D}}$ and from the naturality of τ in A, it follows that $\tau_A = \tau_B \circ T(f)$ for each morphism $f : A \to B$ in $\mathcal{D}$. Sometimes we write $L = \varinjlim T$ and we call the object L of $\mathcal{C}$ a colimit of the functor T.

(d) The category $\mathcal{C}$ is *cocomplete* if every covariant functor from a small category to $\mathcal{C}$ has a colimit.

Now let $\mathcal{C}$ be a category, let I be a directed set and let $\mathcal{D} = \mathrm{Ord}\, I$. Then we say

(a) Any covariant functor $T : \mathcal{D} \to \mathcal{C}$ is called an *inductive system in* $\mathcal{C}$ *over* I and a colimit of the functor T is called an *inductive limit* of the inductive system $(T(i), f_{ij})_I$ and it is denoted by $\varinjlim_{i \in I} T(i)$.

(b) Any covariant functor $S : \mathcal{D}^* \to \mathcal{C}$ is called a *projective system in $\mathcal{C}$ over I* and a limit of the functor S is called a *projective limit* of the projective system $(S(i),\, g_{ij})_I$ and it is denoted by $\lim\limits_{\substack{\leftarrow \\ i \in I}} S(i)$.

Finally, let $\mathcal{B}$ and $\mathcal{C}$ be categories, let T be a contravariant functor from $\mathcal{B}$ to $\mathcal{C}$ and let $\mathcal{D}$ be a small category. We say that

(a) The functor T carries *colimits* of functors from $\mathcal{D}$ into limits if the following condition is satisfied: If $F : \mathcal{D} \to \mathcal{B}$ is a covariant functor and B together with the morphisms $u_j : F(j) \to B$ (where $j \in \operatorname{Ob} \mathcal{D}$) is a colimit of F, then $T(B)$ together with the morphisms $T(u_j) : T(B) \to (T \circ F)(j)$ is a limit of the contravariant functor $T \circ F : \mathcal{D} \to \mathcal{C}$.

(b) The functor T is *continuous* if it carries colimits of functors from any small category into limits.

3. The Category of Effect Algebras

Let L be a set containing at least two distinct elements 0 and 1 called the zero and the unit, let R be a binary relation on L and let $\oplus$ be a map from R to L. The algebraic system $(L,\, R,\, \oplus,\, 0,\, 1)$ is said to be an effect albegra if the following axioms hold:

(*Commutative Law*) If $a, b \in L$ and aRb, then bRa and $a \oplus b = b \oplus a$.

(*Associative Law*) If $a, b, c \in L$, aRb and $(a \oplus b)Rc$, then bRc, $aR(b \oplus c)$ and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

(*Orthosupplementation Law*) For every $a \in L$ there exists a unique $b \in L$ such that aRb and $a \oplus b = 1$.

(*Zero-Unit Law*) If $a \in L$ and $aR1$, then $a = 0$.

We note that the orthosupplementation law implies that R is not empty. If the hypotheses of the associative law are verified, we write $a \oplus b \oplus c$ for the element $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ in L. If $a \in L$, then the unique $b \in L$ such that aRb and $a \oplus b = 1$ is called the *orthosupplement* of a and denoted by a'. If $a \in L, a \neq 0$ and aRa, then a is called an *isotropic element* of L.

An *orthoalgebra* is an effect algebra $(L, R, \oplus, 0, 1)$ in which the zero-unit law is replaced by the stronger

(*Consistence Law*) If $a \in L$ and aRa, then $a = 0$.

Clearly, an effect algebra is an orthoalgebra if and only if it contains no isotropic elements.

If $(L, R, \oplus, 0, 1)$ is an effect algebra, and following the tradition, we denote the binary relation R on L by $\perp$ and if $a, b \in L$ and $a \perp b$ we say that the elements a and b are *orthogonal*.

For a list of nice examples of effect algebras, we refer to [10, 4. Examples] and for some of its properties we refer also to [3] and [9].

Theorem 3.1. *Consider a system* $\mathcal{A}$ *consisting of a class* $\mathrm{Ob}\,\mathcal{A}$ *of all effect algebras, together with a class* $\mathrm{Mor}\,\mathcal{A}$ *which is a disjoint union of the form* $\mathrm{Mor}\,\mathcal{A} = \bigcup \{[L_1, L_2] : L_1, L_2 \in \mathrm{Ob}\,\mathcal{A}\}$ *where* $[L_1, L_2]$ (*with* $L_i = (L_i, \perp_i, \oplus_i, 0_i, 1_i), i = 1, 2)$ *is the class of all map* $f : L_1 \to L_2$ *satisfying the following conditions*:

(a) $f(1_1) = 1_2$.

(b) *If* $a, b \in L_1$, *then* $a \perp_1 b \Leftrightarrow f(a) \perp_2 f(b)$.

(c) *If* $a, b \in L_1$ *and* $a \perp_1 b$, *then* $f(a \oplus_1 b) = f(a) \oplus_2 f(b)$.

Then $\mathcal{A}$ *is a category.*

Proof. It suffices to note that

(1) Since $[L_1, L_2]$ is a subclass of the set of all maps from L_1 to L_2, it follows that $[L_1, L_2]$ is a set.

(2) If $f \in [L_1, L_2]$ and $g \in [L_2, L_3]$, then $g \circ f[L_1, L_3]$.

(3) For all $L \in \mathrm{Ob}\,\mathcal{A}$, the identity map of L is an element of $[L, L]$ satisfying trivially the axiom (iv) of a category.

$\square$

Remark 3.2. (1) If $f : L_1 \to L_2$ is a morphism in the category $\mathcal{A}$, then it can be shown that $f(0_1) = 0_2$ and $f(a') = (f(a))'$ for all $a \in L_1$, where 0 denotes the orthosupplementation in L_1 and L_2.

(2) Let $L = \{0, 1\}$, $\perp = \{(0, 0), (0, 1), (1, 0)\}$ and let us define $0 \oplus 0 = 0$, $0 \oplus 1 = 1$ and $1 \oplus 0 = 1$. Then $(L, \perp, \oplus, 0, 1)$ is an initial object of the category $\mathcal{A}$.

(3) It is easy to see that the category of orthoalgebras is a full subcategory of $\mathcal{A}$.

Theorem 3.3. *The category* $\mathcal{A}$ *has infinite direct sums.*

Proof. Let $(L_j)_{j \in J}$ be an arbitrary family in $\mathrm{Ob}\,\mathcal{A}$. Write $L_j = (L_j, \perp_j, \oplus_j, 0_j, 1_j)$ for all $j \in J$. Put $D = \bigcup_{j \in J} L_j \times \{j\}$ and, for every $j \in J$, we denote by $i_j : L_j \to D$ the map given by the rule $i_j(a) = (a, j)$ where $a \in L_j$. Take

$$S = \{((0_j, j), (0_k, k)) : j, k \in J\} \cup \{((1_j, j), (1_k, k)) : j, k \in J\}$$

and let $\sim$ be the smallest equivalence relation on D containing S.

Write

$$U = D/\sim$$

and denote by p the canonical projection from D onto U. For every element $x \in D$ we denote by $[x]$ its corresponding $\sim$-equivalence class. Since $(0_j, j) \sim (0_k, k)$ and $(1_j, j) \sim (1_k, k)$ for all $j, k \in J$, we can define $0 = [(0_j, j)]$ and $1 = [(1_j, j)]$ for all $j \in J$. Take $\perp$ the binary relation on U defined as follows:

$$[(a, j)] \perp [(b, k)] \text{ if } j = k \text{ and } a \perp_j b.$$

Define a map $\oplus$ from $\perp$ to U by the rule

$$[(a, j)] \oplus [(b, j)] = [(a \oplus_j b, j)].$$

We shall show that $(U, \perp, \oplus, 0, 1)$ is an object of the category $\mathcal{A}$. It is clear that $\oplus$ satisfies the commutative law and the associative law. If we define

$$[(a, j)]' = [(a', j)]$$

for any element of U, it is easy to see that the orthosupplementation law is satisfied. Finally, if $[(a, j)] \perp 1$, then $a \perp_j 1_j$ and therefore $a = 0_j$. So $[(a, j)] = 0$.

For every $j \in J$, let $u_j = p \circ i_j$ and let $u = (u_j : L_j \to U)_{j \in J}$. Consider the pair (U, u) where the first component is an object of $\mathcal{A}$ and the second is a morphism from $L = (L_j)_{j \in J}$ to $\Delta(U)$ in the category $[\mathcal{D}, \mathcal{A}]$ with $\mathcal{D} = \operatorname{Disc} J$.

It remains to show that (U, u) is a universal morphism from L to Δ. Let (M, v) be a pair consisting of an object M of $\mathcal{A}$ and a morphism $v = (v_j : L_j \to M)_{j \in J}$ from L to $\Delta(M)$ in the category $[\mathcal{D}, \mathcal{A}]$. Let $x \in D$. Then there exists a unique pair (a, j) with $j \in J$ and $a \in L_j$ such that $x = (a, j)$. Put $f(x) = v_j(a)$. Then f is a well-defined map from D to M.

We shall show that $x_1, x_2 \in D$ and $x_1 \sim x_2$ imply $f(x_1) = f(x_2)$. Let

$$S_1 = \{(x, y) \in D \times D : f(x) = f(y)\}.$$

Then S_1 is an equivalence relation on D. Since

$$f((0_j, j)) = v_j(0_j) = 0_M = v_k(0_k) = f((0_k, k))$$

and

$$f((1_j, j)) = v_j(1_j) = 1_M = v_k(1_k) = f((1_k, k))$$

for all $j, k \in J$, it follows that S_1 contains S. So $S_1 \supseteq \sim$. Let $x_1, x_2 \in D$ be such that $x_1 \sim x_2$. Then $x_1 S_1 x_2$ and therefore $f(x_1) = f(x_2)$.

Hence there exists a unique map $h : U \to M$ such that $f = h \circ p$. Since $f \circ i_j = v_j$ for all $j \in J$, we get $h \circ u_j = v_j$ for all $j \in J$. A straightforward calculation establishes that h is a morphism in $\mathcal{A}$. Moreover, since $\Delta(h) : \Delta(U) \to \Delta(M)$ is a morphism in the category $[\mathcal{D}, \mathcal{A}]$ such that $\Delta(h)(j) = h$ for all $j \in J$, it follows that $\Delta(h) \circ u = v$. Finally, suppose that there exists another morphism $h' : U \to M$ in $\mathcal{A}$ such that $\Delta(h') \circ u = v$. Then $h' \circ u_j = v_j$ for all $j \in J$. Let $\bar{x} \in U$. Then there exist $j \in J$ and $a \in L_j$ such that $\bar{x} = [(a, j)]$. So

$$h(\bar{x}) = (h \circ u_j)(a) = v_j(a) = (h' \circ u_j)(a) = h'(\bar{x})$$

and therefore $h = h'$.

Remark 3.4. (1) The object U of the category $\mathcal{A}$ is denoted by $\coprod_{j\in J} L_j$ and called the *horizontal sum* of the family $(L_j)_{j\in J}$ following [9].

(2) The family $u = \left(u_j : L_j \to \coprod_{j\in J} L_j\right)_{j\in J}$ of morphisms in $\mathcal{A}$ satisfies the following condition: If $\bar{x}$ and $\bar{y}$ are two orthogonal elements of

$\coprod_{j \in J} L_j$, then there exist $j \in J$ and $a, b \in L_j$ such that $a \perp_j b$, $u_j(a) = \bar{x}$ and $u_j(b) = \bar{y}$.

Theorem 3.5. *The category $\mathcal{A}$ has a coequalizer for every pair of morphisms.*

Proof. Let $f_1, f_2 : L \to M$ be two morphisms in $\mathcal{A}$. Write $M = (M, \perp, \oplus, 0, 1)$. Consider the binary relation on $M : S = \{(f_1(a), f_2(a)) : a \in L\}$ and let $\sim$ be the smallest equivalence relation on M containing S. For $b \in M$ we denote by $\overline{b}$ its corresponding $\sim$ -equivalence class.

Let

$$\overline{M} = M/\sim$$

and let $p : M \to \overline{M}$ be the canonical projection.

Define

$$\overline{\perp} = \{(\overline{b_1}, \overline{b_2}) : b_1, b_2 \in M \text{ and } b_1 \perp b_2\}$$

and define the map $\overline{\oplus} : \overline{\perp} \to \overline{M}$ by the rule

$$\overline{b_1} \mathbin{\overline{\oplus}} \overline{b_2} = \overline{b_1 \oplus b_2}.$$

Then it is easy to verify that $\overline{M} = (\overline{M}, \overline{\perp}, \overline{\oplus}, \overline{0}, \overline{1})$ is an object of $\mathcal{A}$ and p is a morphism in $\mathcal{A}$ which is a surjective map.

It remains to show that the pair $(\overline{M}, p)$ is a coequalizer for f_1 and f_2. Since $S \subseteq \sim$ it follows that $p \circ f_1 = p \circ f_2$. Let $g : M \to N$ be a morphism in $\mathcal{A}$ with $g \circ f_1 = g \circ f_2$. Let $S_1 = \{(b_1, b_2) : b_1, b_2 \in M$ and $g(b_1) = g(b_2)\}$. Then S_1 is an equivalence relation on M containing S. So $S_1 \supseteq \sim$, and therefore $b_1, b_2 \in M$ and $b_1 \sim b_2$ imply $g(b_1) = g(b_2)$. Then there exists a unique map $g' : \overline{M} \to N$ such that $g = g' \circ p$. The verification that g' is a morphism in $\mathcal{A}$ is straightforward. □

Remark 3.6. Since in the pair $(\overline{M},\ p)$ the map p is surjective, it follows from the dual statement of [4, Proposition 2.11] that $(N,\ h)$ is a coequalizer of two morphisms in $\mathcal{A}$, then h is a surjective map.

Theorem 3.7. *The category* $\mathcal{A}$ *is cocomplete.*

Proof. Let T be a covariant functor from a small category $\mathcal{D}$ to $\mathcal{A}$. By Theorems 3.3 and 3.5, $\mathcal{A}$ has infinite direct sums and any pair of morphisms in $\mathcal{A}$ have a coequalizer. Then it follows from the dual statement of [14, Theorem 1, p. 109] that the functor T has a colimit. □

Corollary 3.8. *Let I be a directed set. Then every inductive system in* $\mathcal{A}$ *over I has an inductive limit.*

Theorem 3.9. *The category* $\mathcal{A}$ *has infinite direct products.*

Proof. Let $(L_j)_{j \in J}$ be an arbitrary family in $\mathrm{Ob}\,\mathcal{A}$. Then write $L_j = (L_j,\ \perp_j,\ \oplus_j,\ 0_j,\ 1_j)$ for every $j \in J$. Consider the Cartesian product $V = \mathsf{X}_{j \in J} L_j$ and, for every $j \in J$, we denote by pr_j the canonical projection from V onto L_j. Define $0 = (0_j)_{j \in J}$ and $1 = (1_j)_{j \in J}$. Take $\perp$ the binary relation on V defined as follows:

$$\perp\, = \{(a,\ b) \times V \times V : \mathrm{pr}_j(a) \perp_j \mathrm{pr}_j(b) \text{ for all } j \in J\}$$

and define a map $\oplus : \perp\, \to V$ by the rule $a \oplus b = (a_j \oplus_j b_j)_{j \in J}$.

We shall show that $V = (V,\ \perp,\ \oplus,\ 0,\ 1)$ is an object of the category $\mathcal{A}$. It is clear that $\oplus$ satisfy the commutative law and the associative law. If, for every element $a = (a_j)_{j \in J}$ of V, we define $a' = (a'_j)_{j \in J}$ it is easy to see that the orthosupplementation law is satisfied. Finally, if $a \in V$ and $a \perp 1$, then $a_j \perp_j 1_j$ for all $j \in J$, and therefore $a_j = 0_j$. So $a = 0$.

Let $v = (\mathrm{pr}_j : V \to L_j)_{j \in J}$. Consider the pair (V, v) where the first component is an object of $\mathcal{A}$ and the second is a morphism from $\Delta(V)$ to $L = (L_j)_{j \in J}$ in the category $[\mathcal{D}, \mathcal{A}]$ with $\mathcal{D} = \mathrm{Disc}\, J$.

It remains to show that (V, v) is a universal morphism from Δ to L. Let (W, w) be a pair consisting of an object W of $\mathcal{A}$ and a morphism $w = (w_j : W \to L_j)_{j \in J}$ from $\Delta(W)$ to L in the category $[\mathcal{D}, \mathcal{A}]$. Let $x \in W$. Then $w_j(x) \in L_j$ for all $j \in J$. Hence an element $y \in V$ is uniquely determined by the condition

$$y = (w_j(x))_{j \in J}.$$

So the correspondence $x \to y$ defines a map $f : W \to V$ such that $\mathrm{pr}_j \circ f = w_j$ for all $j \in J$. A straightforward calculation establishes that f is a morphism in $\mathcal{A}$. Since $\Delta(f) : \Delta(W) \to \Delta(V)$ is a morphism in the category $[\mathcal{D}, \mathcal{A}]$ such that $\Delta(f)(j) = f$ for all $j \in J$, it follows that $v \circ \Delta(f) = w$.

Finally, to show the uniqueness of the morphism f, suppose that there exists another morphism $f' : W \to V$ in $\mathcal{A}$ such that $v \circ \Delta(f') = w$. Then $\mathrm{pr}_j \circ f' = w_j$ for all $j \in J$, and therefore $\mathrm{pr}_j \circ f' = \mathrm{pr}_j \circ f$ for all $j \in J$. So $f' = f$.

Remark 3.10. The object V of the category $\mathcal{A}$ is denoted by $\prod_{j \in J} L_j$ and called the *Cartesian product* of the family $(L_j)_{j \in J}$ following [9].

Let $L = (L, \perp, \oplus, 0, 1)$ be an object of $\mathcal{A}$ and let K be a subset of L. We say that K is a *sub-effect algebra* of L if the following axioms hold:

(i) $0, 1 \in K$.

(ii) If $a \in K$, then $a' \in K$.

(iii) If $a, b \in K$ and $a \perp b$, then $a \oplus b \in K$.

Clearly, K in the induced structure is an effect algebra in its own right, so an object of $\mathcal{A}$.

Theorem 3.11. *The category* $\mathcal{A}$ *has an equalizer for every pair of morphisms.*

Proof. Let $f_1, f_2 : L_1 \to L_2$ be two morphisms in $\mathcal{A}$. Then write $L_i = (L_i, \perp_i, \oplus_i, 0_i, 1_i)$ for $i = 1, 2$. Let $K = \{a \in L_1 : f_1(a) = f_2(a)\}$ and let $k : K \to L_1$ be the inclusion map.

We shall show that K is a sub-effect algebra of L_1. Since $f_1(0_1) = 0_2 = f_2(0_1)$ and $f_1(1_1) = 1_2 = f_2(1_1)$, we have that $0_1, 1_1 \in K$. Let $a \in K$. Since $f_1(a') = (f_1(a))' = (f_2(a))' = f_2(a')$, we get $a' \in K$. Let $a, b \in K$ be such that $a \perp_1 b$. Then $f_1(a \oplus_1 b) = f_1(a) \oplus_2 f_1(b) = f_2(a) \oplus_2 f_2(b) = f_2(a \oplus_1 b)$ and therefore $a \oplus_1 b \in K$. Hence K is an object of $\mathcal{A}$ and clearly k is a morphism in $\mathcal{A}$.

It remains to show that the pair (K, k) is an equalizer for f_1 and f_2. We have trivially that $f_1 \circ k = f_2 \circ k$. Let $f : M \to L_1$ be a morphism in $\mathcal{A}$ such that $f_1 \circ f = f_2 \circ f$. Let $a \in M$. Since $f(a) \in L_1$ and $f_1(f(a)) = f_2(f(a))$, we have $f(a) \in K$. Then the rule $a \to f(a)$ defines a morphism $f' : M \to K$ in $\mathcal{A}$. Since $k(f'(a)) = k(f(a)) = f(a)$ for all $a \in M$, we have $k \circ f' = f$. Finally, the uniqueness of f' is trivial.

Theorem 3.12. *The category* $\mathcal{A}$ *is complete.*

Proof. Let T be a covariant functor from a small category $\mathcal{D}$ to $\mathcal{A}$. By Theorems 3.9 and 3.11, $\mathcal{A}$ has infinite direct products and any pair of morphisms in $\mathcal{A}$ have an equalizer. Then it follows from [14, Theorem 1, p. 109] that the functor T has a limit.

Corollary 3.13. *Let I be a directed set. Then every projective system in* $\mathcal{A}$ *over I has a projective limit.*

4. The Category of Base Normed Spaces

Henceforth, V denotes a real vector space.

By a *cone* in V is meant a non-empty subset C such that

$$C + C \subseteq C, \ \mathbb{R}_+ C \subseteq C \quad \text{and} \quad C \cap (-C) = \{0\}.$$

Each cone C in V is clearly convex, contains 0 and defines a partial ordering on V by

$$x \leq_C y \Leftrightarrow y - x \in C.$$

This partial ordering is compatible with the vector structure, that is, if $x \leq_C y$, then $x + z \leq_C y + z$ and $tx \leq_C ty$ for all $z \in Z$ and all $t \in \mathbb{R}_+$.

For short, we call such a partial ordering a vector ordering of V, and the pair $(V, \leq_C)$ an *ordered vector space*. Conversely, if $\leq$ is a vector ordering of V and if we define $C = \{x \in V : 0 \leq x\}$, then C is a cone in V, and $\leq$ is exactly the vector ordering $\leq_C$.

A non-empty subset M of V is said to be *absolutely convex* if M is convex and $tM \subseteq M$ whenever $t \in \mathbb{R}$ and $|t| \leq 1$. If A is a subset of V, the smallest absolutely convex set containing A, denoted acon(A), is called the *absolutely convex hull* of A. Since $\{0\}$ is absolutely convex, it follows that $\operatorname{acon}(\varnothing) = \{0\}$. It can be shown that

$$\operatorname{acon}(A) = \left\{ \sum_{i=1}^{m} t_i x_i : x_i \in A, \ t_i \in \mathbb{R}, \ \sum_{i=1}^{m} |t_i| \leq 1, \ n \in \mathbb{N} \right\}$$

(see [13, pp. 160 and 173-174]).

A non-empty subset B of V is said to be *absorbing* if, for every $x \in V$, there exists a real number $s > 0$ such that $x \in tB$ for all $t \in \mathbb{R}$ with $|t| \geq s$.

If M is an absorbing and absolutely convex subset of V, then the

Minkowski functional $\rho_M : V \to \mathbb{R}_+$ on M defined by

$$\rho_M(x) = \inf\{t > 0 : x \in tM\}$$

is a semi-norm on V such that

$$\{x \in V : \rho_M(x) < 1\} \subseteq M \subseteq \{x \in V : \rho_M(x) \leq 1\}$$

(see [18, p. 40]).

A subset K of V is called a *cone base* of V if K is convex and any relation of the form $sx = ty$ with $x, y \in K$ and $s, t \in \mathbb{R}_+$ implies $s = t$. Clearly $\varnothing$ is trivially a cone base of V and $0 \notin K$ for all cone base K of V. The name is justified for the following immediate fact: If K is a cone base of V, then $\mathbb{R}_+ K \bigcup \{0\}$ is a cone in V.

An *affine subspace* of V is a translation of a subspace of V, that is, every set M of the form $M = x + W$, where $x \in V$ and W is a subspace of V. If A is a subset of V, the smallest affine subspace of V containing A, denoted by aff(A), is called the *affine hull* of A. Since for every $x \in V$, the set $\{x\}$ is an affine subspace of V, it follows that $\operatorname{aff}(\varnothing) = \varnothing$. It can be shown that

$$\operatorname{aff}(A) = \left\{ \sum_{i=1}^{n} t_i x_i : x_i \in A, t_i \in \mathbb{R}, \sum_{i=1}^{n} t_i = 1, n \in \mathbb{N} \right\}$$

(see [6, pp. 21-23]).

Lemma 4.1. *Let K be a convex subset of V. Then K is a cone base of V if and only if* $0 \notin \operatorname{aff}(K)$.

Proof. The Lemma is trivial if $K = \varnothing$. It remains to prove the case $K \neq \varnothing$.

Assume that K is a cone base of V and $0 \in \operatorname{aff}(K)$. Then there exist a finite sequence $(x_i)_{1 \leq i \leq n}$ of vectors in K and a finite sequence $(t_i)_{1 \leq i \leq n}$ of nonzero real numbers such that $\sum_{i=1}^{n} t_i = 1$ and $\sum_{i=1}^{n} t_i x_i = 0$. Then, since

$0 \notin K$ and K is convex, it follows that $n > 1$ and $\{i \in \{1, 2, ..., n\} : t_i < 0\} \neq \emptyset$. So there exist an integer m and a permutation σ of the set $\{1, 2, ..., n\}$ such that $1 \leq m \leq n$, $t_{\sigma(i)} > 0$ for $i = 1, 2, ..., m$ and $t_{\sigma(i)} < 0$ for $i = m+1, m+2, ..., n$. Then

$$\sum_{i=1}^{m} t_{\sigma(i)} x_{\sigma(i)} + \sum_{i=m+1}^{n} t_{\sigma(i)} x_{\sigma(i)} = 0.$$

Write

$$s = \sum_{i=1}^{m} t_{\sigma(i)},\ t = \sum_{i=m+1}^{n} t_{\sigma(i)},\ u = \sum_{i=1}^{m} \frac{t_{\sigma(i)}}{s} x_{\sigma(i)} \text{ and } v = \sum_{i=m+1}^{n} \frac{t_{\sigma(i)}}{t} x_{\sigma(i)}.$$

Then $s > 0$, $t < 0$, $s + t = 1$, $u, v \in K$ and $su + tv = 0$. So $su = (-t)v$, and since K is a cone base, it follows that $s + t = 0$, in contradiction with $s + t = 1$.

Assume that $0 \notin \text{aff}(K)$ and there exist $x, y \in K$ and $s, t \in \mathbb{R}$ such that $s > 0$, $t > 0$, $s \neq t$ and $sx = ty$. Hence $0 = \frac{s}{s-t}x + \frac{-t}{s-t}y \in \text{aff}(K)$, in contradiction with the hypothesis.

Corollary 4.2. *Let K be a cone base of V, let $(x_i)_{1 \leq i \leq n}$ be a finite sequence of vectors in K and let $(t_i)_{1 \leq i \leq n}$ be a finite sequence of nonzero real numbers. If $\sum_{i=1}^{n} t_i x_i = 0$, then $\sum_{i=1}^{n} t_i = 0$.*

Proof. Suppose that $t = \sum_{i=1}^{n} t_i \neq 0$. Then $\sum_{i=1}^{n} \frac{t_i}{t} x_i \in \text{aff}(K)$, and therefore $0 \in \text{aff}(K)$, in contradiction with Lemma 4.1.

Corollary 4.3. *Let K be a cone base of V and let $C = \mathbb{R}_+ K \cup \{0\}$. Then any relation of the form $sx \leq_C ty$ with $x, y \in K$ and $s, t \in \mathbb{R}$ implies $s \leq t$.*

Proof. We may suppose that $K \neq \varnothing$.

Since $u \leq_C v \Leftrightarrow v - u \in C$ for all $u, v \in V$, it follows that $ty - sx \in \mathbb{R}_+K$. So there exist $r \in \mathbb{R}_+$ and $z \in K$ such that $ty - sx = rz$. Then $ty - sx - rz = 0$, and by Corollary 4.2 it follows that $t - s - r = 0$, and therefore $t = s + r \geq s$. □

Lemma 4.4. *Let K be a non-empty convex subset of V. Then* $\operatorname{acon}(K) = \{tx - (1-t)y : x, y \in K \text{ and } t \in [0, 1]\}$.

Proof. Write $M = \{tx - (1-t)y : x, y \in K \text{ and } t \in [0, 1]\}$. Since $K \cup (-K) \subseteq \operatorname{acon}(K)$, it follows that $M \subseteq \operatorname{acon}(K)$.

The convexity of M follows from the convexity of K by a straightforward argument. We shall show that M is absolutely convex. Let $z \in M$ and let $s \in \mathbb{R}$ be such that $|s| \leq 1$. Since $M = -M$, we may suppose that $0 < s \leq 1$. Write $z = tx - (1-t)y$ for $s, y \in K$ and $t \in [0, 1]$. Since $sx = \frac{1+s}{2}x - \frac{1-s}{2}x$, $-sy = \frac{1-s}{2}y - \frac{1+s}{2}y$, $\frac{1+s}{2}$, $\frac{1-s}{2} \in \mathbb{R}_+$ and $\frac{1+s}{2} + \frac{1-s}{2} = 1$, it follows that $sx, -sy \in M$ and $sz = t(sx) + (1-t)(-sy) \in M$ because M is convex. Hence $sM \subseteq M$.

Since $K \subseteq M$, we get $\operatorname{acon}(K) \subseteq M$. □

If K is a non-empty convex subset of V, the set $\mathbb{R}_+K - \mathbb{R}_+K$ is a subspace of V called the *linear hull* of K and denoted by lin(K). A cone C in V is said to be *generating* if $V = C - C$. Clearly, if C is a generating cone in V, then $V = \operatorname{lin}(C)$.

Lemma 4.5. *Let K be a cone base of V such that* $V = \operatorname{lin}(K)$. *Then* $\operatorname{acon}(K)$ *is an absorbing set.*

Proof. Let $z \in V$. Then there exist $x, y \in K$ and $s, t \in \mathbb{R}_+$ such that $z = sx - ty$. We may suppose that $s + t > 0$. Let $r \in \mathbb{R}$ be such that

$|r| \geq s + t$. Since $\frac{z}{r} = \frac{s}{r}x + \frac{-t}{r}y$ and $\left|\frac{s}{r}\right| + \left|\frac{-t}{r}\right| = \frac{s + t}{|r|} \leq 1$, it follows that $\frac{z}{r} \in \operatorname{acon}(K)$, and therefore $z \in r\operatorname{acon}(K)$. $\square$

Remark 4.6. Let K be a cone base of V such that $V = \operatorname{lin}(K)$. If $M = \operatorname{acon}(K)$, then the Lemma 4.5 allows us to define the Minkowski functional ρ_M on M which is a semi-norm on V.

Theorem 4.7. *Let K be a cone base of V such that $V = \operatorname{lin}(K)$. If $M = \operatorname{acon}(K)$, then*

$$\rho_M(z) = \{s + t : z = sx - ty,\ x,\ y \in K \text{ and } s,\ t \in \mathbb{R}_+\}$$

for all $z \in V$.

Proof. Since $K \subseteq \operatorname{acon}(K)$, it follows that $\rho_M(x) \leq 1$ for all $x \in K$. Let $z \in V$. We can write $z = sx - ty$ with $x, y \in K$ and $s, t \in \mathbb{R}_+$. Since ρ_M is a semi-norm on V, it follows that $\rho_M(z) \leq s\rho_M(x) + t\rho_M(y) \leq s + t$. Hence $\rho_M(z)$ is a lower bound of the set

$$N = \{s + t : z = sx - ty,\ x,\ y \in K \text{ and } s,\ t \in \mathbb{R}_+\}.$$

Let $\varepsilon > 0$. By definition of $\rho_M(z)$ there exists $r > 0$ such that $z \in rM$ and $r > \rho_M(z) + \varepsilon$. By Lemma 4.4, there exist $x, y \in K$ and $t \in [0, 1]$ such that $\frac{z}{r} = tx - (1 - t)y$. So $z = (rt)x - ((1 - t)r)y$ with $rt + (1 - t)r = r$, and therefore $r \in N$. Hence $\rho_M(z) = \inf N$.

A pair (V, K) where K is a cone base of V such that $V = \operatorname{lin}(K)$ is called a *base normed space* or a BN-space if the Minkowski functional on acon(K) is a norm, referred to as the *base norm* on V and denoted by $\|\cdot\|_V$.

Two important examples of BN-spaces are the following:

(1) Recall that a normed Riesz space $(V, \leq, \|\cdot\|)$ is said to be an AL-space if $\| x + y \| = \| x \| + \| y \|$ for all $x, y \in V$ with $0 \leq x, y$.

If $(V, \leq, \|\cdot\|)$ is an AL-space and if $K = \{x \in V : 0 \leq x \text{ and } \| x \| = 1\}$, then the pair (V, K) is a BN-space with $\|\cdot\|_V = \|\cdot\|$.

(2) Let A be a C^*-algebra with unit 1 and let A^* the Banach dual space. Consider the self-adjoint part of A^*:

$$A_{sa}^* = \{f \in A^* : f(a) = \overline{f(a^*)} \text{ for all } a \in A\}$$

and the state space of A:

$$S(A) = \{f \in A^* : \| f \| = 1 = f(1)\}.$$

Then the pair $(A_{sa}^*, S(A))$ is a BN-space with the given norm on A^* restricted to A_{sa}^* as the base norm.

Theorem 4.8. *Consider a system* $\mathcal{B}$ *consisting of a class* $\mathrm{Ob}\,\mathcal{B}$ *of all BN-spaces, together with a class* $\mathrm{Mor}\,\mathcal{B}$ *which is a disjoint union of the form*

$$\mathrm{Mor}\,\mathcal{B} = \bigcup \{[(V_1, K_1), (V_2, K_2)] : (V_i, K_i) \in \mathrm{Ob}\,\mathcal{B} \text{ for } i = 1, 2\}$$

where $[(V_1, K_1), (V_2, K_2)]$ *is the class of all linear maps* $f : V_1 \to V_2$ *such that* $f(K_1) \subseteq K_2$.

Then $\mathcal{B}$ *is a category.*

Proof. It suffices to note that

(1) Since $[(V_1, K_1), (V_2, K_2)]$ is a subclass of the set of all maps from V_1 to V_2, it follows that $[(V_1, K_1), (V_2, K_2)]$ is a set.

(2) If $f \in [(V_1, K_1), (V_2, K_2)]$ and $g \in [(V_2, K_2), (V_3, K_3)]$, then $g \circ f \in [(V_1, K_1), (V_3, K_3)]$.

(3) For all $(V, K) \in \operatorname{Ob} \mathcal{B}$, the identity map of V is an element of $[(V, K), (V, K)]$ satisfying trivially the axiom (iv) of a category.

Remark 4.9. We note that $(\{0\}, \varnothing)$ is an initial object of $\mathcal{B}$.

Let V_1 and V_2 be two real vector spaces, let K_1 be a convex subset of V_1 and let K_2 be a convex subset of V_2. A map $f : K_1 \to K_2$ is called affine if $f(tx + (1-t)y) = tf(x) + (1-t)f(y)$ for all $x, y \in K_1$ and all $t \in [0, 1]$.

Lemma 4.10. *Let (V_1, K_1) and (V_2, K_2) be two objects of $\mathcal{B}$ and let $f : K_1 \to K_2$ be an affine map. Then there exists a unique element $f' \in [(V_1, K_1), (V_2, K_2)]$ such that $f'|_{K_1} = f$.*

Proof. Let $z \in V_1$. Then there exist $x, y \in K_1$ and $s, t \in \mathbb{R}_+$ such that $z = sx - ty$. Put $f'(z) = sf(x) - tf(y)$. We shall show that f' is a well-defined map from V_1 to V_2. Suppose that $z = s_1x_1 - t_1y_1$ with $x_1, y_1 \in K_1$ and $s_1, t_1 \in \mathbb{R}_+$. Then $sx - ty - s_1x_1 + t_1y_1 = 0$, and from Corollary 4.2 it follows that $s - t - s_1 + t_1 = 0$. So $s + t_1 = s_1 + t$. If $s + t_1 = 0$, then s, t, s_1 and t_1 are equal to zero and trivially $sf(x) - tf(y) = s_1f(x) - t_1f(y)$. If $s + t_1 \neq 0$, then

$$\frac{s}{s+t_1}x + \frac{t_1}{s+t_1}y_1 = \frac{s_1}{s+t_1}x_1 + \frac{t}{s+t_1}y \in K_1$$

and therefore

$$\frac{s}{s+t_1}f(x) + \frac{t_1}{s+t_1}f(y_1) = \frac{s_1}{s+t_1}f(x_1) + \frac{t}{s+t_1}f(y).$$

Hence $sf(x) - tf(y) = s_1f(x_1) - t_1f(y_1)$.

The verification that f' is linear is straightforward. Since $z = 1 \cdot z - 0 \cdot 0$ for all $z \in K_1$ it follows that $f'|_{K_1} = f$. The uniqueness of f' is trivial.

Theorem 4.11. *The category $\mathcal{B}$ has infinite direct products.*

Proof. Let $((V_j, K_j))_{j\in J}$ be an arbitrary family in $\mathrm{Ob}\,\mathcal{B}$. Consider the real vector space given by the Cartesian product $W = \mathsf{X}_{j\in J} V_j$ with canonical projections $\mathrm{pr}_j : W \to V_j$ for all $j \in J$. Define $K = \{x \in W :$ $\mathrm{pr}_j(x) \in K_j$ for all $j \in J\}$.

We shall show that K is a cone base of W. We may suppose that $K \neq \varnothing$. Then $K_j \neq \varnothing$ for all $j \in J$. The linearity of pr_j and the convexity of K_j for all $j \in J$ imply that K is convex. Assume that $sx = ty$ with $x, y \in K$ and $s, t \in \mathbb{R}_+$. Let $j \in J$. Then $s\,\mathrm{pr}_j(x) = t\,\mathrm{pr}_j(y)$ and since K_j is a cone base of V_j, we get $s = t$.

Now define $V = \mathrm{lin}(K)$. Then V is a subspace of W with cone base K. To show that (V, K) is an object of $\mathcal{B}$ it remains to verify that $\|\cdot\|_V$ $= \rho_{\mathrm{acon}(K)}$ is a norm on V. Let $z \in V$ be such that $\|z\|_V = 0$ and let $\varepsilon > 0$. By Theorem 4.7 there exist $x, y \in K$ and $s, t \in \mathbb{R}_+$ such that $z = sx - ty$ and $s + t < \varepsilon$. Let $j \in J$. Then $\mathrm{pr}_j(z) = s\,\mathrm{pr}_j(x) - t\,\mathrm{pr}_j(y)$. Since $\mathrm{pr}_j(x)$, $\mathrm{pr}_j(y) \in K_j$, it follows that $\|\mathrm{pr}_j(z)\|_{V_j} < s + t < \varepsilon$. So $\mathrm{pr}_j(z) = 0$ for all $j \in J$, and therefore $z = 0$.

Define $v_j = \mathrm{pr}_j|_V$ for all $j \in J$. Clearly, $v_j \in [(V, K), (V_j, K_j)]$ for all $j \in J$. Write $v = (v_j : (V, K) \to (V_j, K_j))_{j\in J}$. Consider the pair $((V, K), v)$ where the first component is an object of $\mathcal{B}$ and the second is a morphism from $\Delta(V, K)$ to $((V_j, K_j))_{j\in J}$ in the category $[\mathcal{D}, \mathcal{B}]$ where $\mathcal{D} = \mathrm{Disc}\,J$.

It remains to show that $((V, K), v)$ is a universal morphism from Δ to $((V_j, K_j))_{j\in J}$. Let $((V', K'), v')$ be a pair consisting of an object (V', K')

of $\mathcal{B}$ and a morphism $v' = (v'_j : (V', K') \to (V_j, K_j))_{j \in J}$. from $\Delta(V', K')$ to $((V_j, K_j))_{j \in J}$ in the category $[\mathcal{D}, \mathcal{B}]$. Let $z \in K'$. Since $v'_j(K') \subseteq K_j$, it follows that $v'_j(z) \in K_j$ for all $j \in J$. Then an element $w \in W$ is uniquely determined by the condition $w = (v'_j(z))_{j \in J}$. Since $\mathrm{pr}_j(w) = v'_j(z) \in K_j$ for all $j \in J$, it follows that $w \in K$ and $\mathrm{pr}_j(w) = v_j(w)$. Then the correspondence $z \to w$ defines a map $f : K' \to K$ such that $v_j(f(z)) = v'_j(z)$ for all $z \in K'$ and all $j \in J$.

We shall show that f is an affine map. Let $z_1, z_2 \in K'$ and let $t \in [0, 1]$. Since

$$\begin{aligned} v_j(f(tz_1 + (1-t)z_2)) &= v'_j(tz_1 + (1-t)z_2) \\ &= tv'_j(z_1) + (1-t)v'_j(z_2) \\ &= tv_j(f(z_1)) + (1-t)v_j(f(z_2)) \\ &= v_j(tf(z_1) + (1-t)f(z_2)) \end{aligned}$$

and $v_j = \mathrm{pr}_j|_V$ for all $j \in J$, it follows that $f(tz_1 + (1-t)z_2) = tf(z_1) + (1-t)f(z_2)$.

By Lemma 4.10, there exists a unique morphism $f' : (V', K') \to (V, K)$ in $\mathcal{B}$ such that $f'|_{K'} = f$. We shall show that $v'_j = v_j \circ f'$ for all $j \in J$. Let $z \in V'$. Then there exist $x, y \in K'$ and $s, t \in \mathbb{R}_+$ such that $z = sx - ty$. Then $v'_j(z) = sv'_j(x) - tv'_j(y) = sv_j(f(x)) - tv_j(f(y)) = v_j(sf(x) - tf(y)) = v_j(f'(sx - ty) = (v_j \circ f'))(z)$ and therefore $v'_j = v_j \circ f'$ for all $j \in J$.

Since $\Delta(f') : \Delta(V', K') \to \Delta(V, K)$ is a morphism in the category $[\mathcal{D}, \mathcal{B}]$ such that $\Delta(f')(j) = f'$ for all $j \in J$, it follows that $v' = v \circ \Delta(f')$ by the definition of composition of natural transformations. Finally, to show the uniqueness of the morphism f' suppose that there exists another morphism $f'' : (V', K') \to (V, K)$ in $\mathcal{B}$ such that $v' = v \circ \Delta(f'')$. Then $v'_j = v_j \circ f''$ for all $j \in J$, and therefore $v_j \circ f' = v_j \circ j''$ for all $j \in J$. Since $v_j = \mathrm{pr}_j|_V$ it follows that $f' = f''$. $\square$

Lemma 4.12. *Let (V, K) be an object of $\mathcal{B}$, let L be a convex subset of K and let $W = \mathrm{lin}(L)$. Then (W, L) is also an object of $\mathcal{B}$.*

Proof. Since K is a cone base of V, it follows that L is also a cone base of V, and therefore a cone base of W. Moreover, $\mathrm{acon}(L) \subseteq \mathrm{acon}(K)$, and therefore $\| x \|_V \le \| x \|_W$ for all $x \in W$. Finally, to prove that $\| \cdot \|_W$ is a norm on W, let $x \in W$ be such that $\| x \|_W = 0$. So $\| x \|_V = 0$, and therefore $x = 0$. $\square$

Theorem 4.13. *The category $\mathcal{B}$ has an equalizer for all pair of morphisms.*

Proof. Let $f_1, f_2 : (V, K) \to (V', K')$ be two morphisms in $\mathcal{B}$. Define $L = \{x \in K : f_1(x) = f_2(x)\}$. It is easy to see that L is a convex subset of K. Let $W = \mathrm{lin}(L)$. By Lemma 4.12, (W, L) is an object of $\mathcal{B}$. Let $i : W \to V$ be the inclusion map. Clearly, $i : (W, L) \to (V, K)$ is a morphism in $\mathcal{B}$.

We shall show that the pair $((W, L), i)$ is an equalizer for f_1 and f_2. It is clear that $f_1 \circ i = f_2 \circ i$. Let $h : (V'', K'') \to (V, K)$ be a morphism in $\mathcal{B}$ with $f_1 \circ h = f_2 \circ h$. Let $z \in K''$. Then $h(x) \in K$ and $f_1(h(x)) = f_2(h(x))$. So $h(x) \in L$. Now let $z \in V''$. Then there exist $x, y \in K''$ and $s, t \in \mathbb{R}_+$ such that $z = sx - ty$. Hence $h(z) = sh(x) - th(y)$. Since $h(x), h(y) \in L$, it follows that $h(z) \in W$. Then we can define a map $h' : V'' \to W$ by the rule $h'(z) = h(z)$. Clearly, $h' : (V'', K'') \to (W, L)$ is a morphism in $\mathcal{B}$ such that $h = i \circ h'$, and the uniqueness of h' is trivial. $\square$

Remark 4.14. Since in the pair $((W, L), i)$ the set map $i : W \to V$ is trivially injective, it follows from [4, Proposition 2.11] that if $((U, M), h)$ is an equalizer of two morphisms in $\mathcal{B}$, then the underlying set map of h is injective.

Theorem 4.15. *The category* $\mathcal{B}$ *is complete.*

Proof. Let T be a covariant functor from a small category $\mathcal{D}$ to $\mathcal{B}$. By Theorems 4.11 and 4.13, $\mathcal{B}$ has infinite direct products and any pair of morphisms in $\mathcal{B}$ have an equalizer. Then it follows from [14, Theorem 1, p. 109] that the functor T has a limit. $\square$

Corollary 4.16. *Let I be a directed set. Then every projective system in* $\mathcal{B}$ *over I has a projective limit.*

5. Probability Measures on an Effect Algebra

Let $L = (L, \perp, \oplus, 0, 1)$ be an effect algebra and let $a, b \in L$. We write $a \leq b$ if there exists an element $c \in L$ such that $a \perp c$ and $a \oplus c = b$. In [9] it is established that the pair $(L, \leq)$ is a partially ordered set with the following properties:

(1) $0 \leq a \leq 1$ for all $a \in L$.

(2) Let $a, b \in L$. Then $a \perp b$ if and only if $a \leq b'$.

(3) If $a, b \in L$ and $a \leq b$, then $b' \leq a'$.

(4) If $a, b \in L$ and $a \leq b$, then $a \perp (a \oplus b')'$ and $b = a \oplus (a \oplus b')'$.

(5) Let $a, b, c \in L$ be such that $a \perp c$ and $b \perp c$. If $a \oplus c = b \oplus c$, then $a = b$.

(6) Let $a, b, c \in L$ be such that $a \perp c$ and $b \perp c$. If $a \oplus c \leq b \oplus c$, then $a \leq b$.

If $a, b \in L$ and $a \leq b$, then the property (5) implies that the element $c \in L$ such that $a \perp c$ and $a \oplus c = b$ is unique. It is denoted by $b - a$ and called the *difference* between b and a. For some properties of differences in effect algebras, we refer to Lemmas 4.7, 4.8 and 4.9 and Remark 4.10 of [9].

Let $L = (L, \perp, \oplus, 0, 1)$ be an effect algebra, let $(a_i)_{i \in I}$ be a finite non-empty family of elements of L and let $n = \mathrm{card}(I)$. From the $n!$ different total orderings on I we choose one of them denoted by $\leq$. The finite family $(a_i)_{i \in I}$ together with the totally ordered set $(I, \leq)$ define an algebraic system called a *suite* in L (of n elements) and it is denoted by $(a_i)_{i \in (I, \leq)}$. A suite in L of n elements $(a_i)_{i \in (I, \leq)}$ is said to be *orthogonal* in L if either $n = 1$ or, when $n \geq 2$, if

$$a_{i_0} \perp a_{i_1} \quad \text{and} \quad a_{i_j} \perp (\cdots((a_{i_0} \oplus a_{i_1}) \oplus a_{i_2}) \cdots \oplus a_{i_{j-1}})$$

for all $2 \leq j \leq n - 1$ where $i_0 = \min I$ and $i_k = \min(I \backslash \{i_0, i_1, \ldots, i_{k-1}\})$ for all $1 \leq k \leq n - 1$. In this case, the element s of L defined respectively by the formulae

$$s = a_{i_0} \quad \text{and} \quad s = (\cdots((a_{i_0} \oplus a_{i_1}) \oplus a_{i_2}) \cdots \oplus a_{i_{n-2}}) \oplus a_{i_{n-1}}$$

is called the $\oplus$*-join* of the orthogonal suite $(a_i)_{i \in (I, \leq)}$ and it is denoted by $\oplus_{i \in I} a_i$.

Lemma 5.1. *Let $(I, \leq)$ be a finite totally ordered set with $\mathrm{card}(I) = n \geq 2$, let $i_0 = \min(I)$ and let $I' = I \backslash \{i_0\}$. If $(a_i)_{i \in (I, \leq)}$ is an orthogonal suite in L of n elements, then $(a_i)_{i \in (I', \leq)}$ is an orthogonal suite of $n - 1$ elements, $a_{i_0} \perp (\oplus_{i \in I'} a_i)$ and $\oplus_{i \in I} a_i = a_{i_0} \oplus (\oplus_{i \in I'} a_i)$.*

Proof. By induction on n. □

Now we fix an effect algebra $L(L, \perp, \oplus, 0, 1)$.

It is well known that the real vector space $(\mathbb{R}^L, +, \cdot)$, with addition and scalar multiplication defined coordinate-wise and endowed with the product topology τ, is a Hausdorff locally convex topological vector space over $\mathbb{R}$. Also, a net $(f_\alpha)_{\alpha \in D}$ in $\mathbb{R}^L$ τ-converges to an element $f \in \mathbb{R}^L$ if and only if $\lim\limits_{\alpha} f_\alpha(a) = f(a)$ for all $a \in L$.

An element $\mu \in \mathbb{R}^L$ is said to be a *measure* μ on L if $\mu(a \oplus b) = \mu(a) + \mu(b)$ whenever $a, b \in L$ and $a \perp b$.

Lemma 5.2. *Let* μ *be a measure on* L. *We have*

(i) $\mu(0) = 0$.

(ii) *If* $a, b \in L$ *and* $a \leq b$, *then* $\mu(b - a) = \mu(b) - \mu(a)$.

(iii) *Let* $(I, \leq)$ *be a finite totally ordered set with* $card(I) = n \geq 2$. *If* $(a_i)_{i \in (I, \leq)}$ *is an orthogonal suite in* L *with* n *elements, then* $\mu(\oplus_{i \in I} a_i) = \sum_{i \in I} \mu(a_i)$.

Proof. (i) It suffices to note that $0 \perp 0$ and $0 \oplus 0 = 0$ by [9, Lemma 2.3 (v)].

(ii) It suffices to note that $a \perp (b - a)$ and $a \oplus (b - a) = b$.

(iii) We proceed by induction on n. For $n = 2$ the formula follows from the definition of measure.

Let $n \in \mathbb{N}$ be such that $n \geq 3$ and suppose that the formula is true for every orthogonal suite in L of $n - 1$ elements. Let $(a_i)_{i \in (I, \leq)}$ be an orthogonal suite in L of n elements. Let $i_0 = \min I$ and let $I' = I \backslash \{i_0\}$. By Lemma 5.1 it follows that $(a_i)_{i \in (I', \leq)}$ is an orthogonal suite in L of $n - 1$ elements, $a_{i_0} \perp (\oplus_{i \in I'} a_i)$ and $\oplus_{i \in I} a_i = a_{i_0} \oplus (\oplus_{i \in I'} a_i)$. Then the induction hypothesis implies that

$$\mu\left(\bigoplus_{i\in I} a_i\right) = \mu(a_{i_0}) + \mu\left(\bigoplus_{i\in I'} a_i\right) = \mu(a_{i_0}) + \sum_{i\in I\setminus\{i_0\}} \mu(a_i) = \sum_{i\in I} \mu(a_i).$$

□

A measure μ on L is said to be *positive* if $\mu(a) \in \mathbb{R}_+$ for all $a \in L$. Trivially, the zero element of $\mathbb{R}^L$ is a positive measure on L.

Lemma 5.3. *Let μ be a positive measure on L. If $a, b \in L$ and $a \le b$, then $\mu(a) \le \mu(b)$. In particular, $0 \le \mu(a) \le \mu(1)$ for all $a \in L$.*

Proof. By the property (4), we have $a \perp (a \oplus b')'$ and $b = a \oplus (a \oplus b')'$. Then $\mu(b) = \mu(a) + \mu((a \oplus b')') \ge \mu(a)$ since μ is a positive measure on L.

□

An element $\mu \in \mathbb{R}^L$ is said to be a *probability measure* on L if μ is a positive measure on L such that $\mu(1) = 1$. We denote by $\Omega(L)$ the set of all probability measures on L. It is clear that $\Omega(L)$ is a convex subset of $\mathbb{R}^L$. For examples of effect algebras L such that $\Omega(L) = \varnothing$ or $\Omega(L)$ is a singleton, see [12].

We note that if $\mu \in \Omega(L)$, then Lemma 5.3 implies $\mu(a') = 1 - \mu(a)$ for all $a \in L$ since $a' = 1 - a$.

Lemma 5.4. *The set $\Omega(L)$ is τ-closed in $\mathbb{R}^L$.*

Proof. Let $(\mu_\alpha)_{\alpha\in D}$ be a net in $\Omega(L)$ which τ-converges to an element $\mu \in \mathbb{R}^L$. Then $\lim_\alpha \mu_\alpha(a) = \mu(a)$ for all $a \in L$. In particular, $\mu(1) = 1$ and $\mu(a) \in \mathbb{R}_+$ for all $a \in L$. Let $a, b \in L$ be such that $a \perp b$. Then $\lim_\alpha \mu_\alpha (a \oplus b) = \mu(a \oplus b)$. Since $\mu_\alpha(a \oplus b) = \mu_\alpha(a) + \mu_\alpha(b)$ for all $\alpha \in D$ and $\lim_\alpha(\mu_\alpha(a) + \mu_\alpha(b)) = \mu(a) + \mu(b)$, it follows that $\mu(a \oplus b) = \mu(a) + \mu(b)$, and therefore $\mu \in \Omega(L)$. □

Corollary 5.5. *The set* $\Omega(L)$ *is* τ*-bounded.*

Proof. Since $\Omega(L) \subseteq [0, 1]^L$ and $[0, L]^L$ is τ-compact by Tychono Theorem, it follows from Lemma 5.4 that $\Omega(L)$ is τ-bounded. □

A measure on L is said to be *Jordan* if it can be represented as a difference of two positive measures on L. We denote by $J(L)$ the vector subspace of $\mathbb{R}^L$ of all Jordan measures on L. It is easy to verify that $\Omega(L)$ is a cone base of $J(L)$.

Theorem 5.6. *The pair* $(J(L), \Omega(L))$ *is a BN-space and its norm topology is finer than the topology* τ.

Proof. It suffices to note that $J(L) = \mathrm{lin}(\Omega(L))$ and to apply Corollary 5.5 and Theorem 2.8(i) of [17, p. 589]. □

For every object $L = (L, \perp, \oplus, 0, 1)$ of the category $\mathcal{A}$ we associate the object $T(L) = (J(L), \Omega(L))$ of the category $\mathcal{B}$.

Lemma 5.7. *Let* $f : L_1 \to L_2$ *be a morphism in* $\mathcal{A}$ *where* $L_i = (L_i, \perp_i, \oplus_i, 0_i, 1_i)$ *for* $i = 1, 2$. *For every* $\mu \in \Omega(L_2)$ *we write* $T(f)(\mu) = \mu \circ f$. *Then*

(a) $T(f)(\mu) \in \Omega(L_1)$.

(b) $T(f) : \Omega(L_2) \to \Omega(L_1)$ *is an affine map.*

Proof. (a) Clearly, $T(f)(\mu) \in \mathbb{R}^{L_1}$, $T(f)(\mu)(a) \in \mathbb{R}_+$ for all $a \in L_1$ and $T(f)(\mu)(1_1) = 1$. Let $a, b \in L_1$ be such that $a \perp_1 b$. Then $T(f)(\mu)(a \oplus_1 b) = (\mu \circ f)(a \oplus_1 b) = \mu(f(a) \oplus_2 f(b)) = \mu(f(a)) + \mu(f(b))$ $= T(f)(\mu)(a) + T(f)(\mu)(b)$, and therefore, $T(f)(\mu) \in \Omega(L_1)$.

(b) Let $\mu_1, \mu_2 \in \Omega(L_2)$ and let $t \in [0, 1]$. For every $a \in L_1$ we have

$$T(f)(t\mu_1 + (1-t)\mu_2)(a) = (t\mu_1 + (1-t)\mu_2)(f(a))$$

$$= t\mu_1(f(a)) + (1-t)\mu_2(f(a))$$

$$= tT(f)(\mu_1)(a) + (1-t)T(f)(\mu_2)(a)$$

and therefore $T(f)(t\mu_1 + (1-t)\mu_2) = tT(f)(\mu_1) + (1-t)T(f)(\mu_2)$. □

Remark 5.8. It follows from Lemmas 4.10 and 5.7 that there exists a unique element of $[(J(L_2), \Omega(L_2)), (J(L_1), \Omega(L_1))]$ whose restriction to $\Omega(L_2)$ is $T(f)$. We denote this extension by the same symbol.

Theorem 5.9. *The prescription T which assigns to each object L of $\mathcal{A}$ the object $T(L) = (J(L), \Omega(L))$ of $\mathcal{B}$ and to each morphism $f : L_1 \to L_2$ in $\mathcal{A}$ the morphism $T(f) : T(L_2) \to T(L_1)$ in $\mathcal{B}$, is a contravariant functor form $\mathcal{A}$ to $\mathcal{B}$.*

Proof. Let L be an object of $\mathcal{A}$. Since $T(1_L)(\mu) = \mu$ for all $\mu \in \Omega(L)$, the extension of $T(1_L)$ to $J(L)$ is trivially $1_{(J(L),\Omega(L))}$. So $T(1_L) = 1_{T(L)}$.

Let $f : L_1 \to L_2$ and $g : L_2 \to L_3$ be two morphisms in $\mathcal{A}$. Then $g \circ f \in [L_1, L_3]_{\mathcal{A}}$. Let $\mu \in \Omega(L_3)$. Since $T(g \circ f)(\mu) = \mu \circ (g \circ f)$ and $(T(f) \circ T(g))(\mu) = T(f)(\mu \circ g) = (\mu \circ g) \circ f$, it follows that $T(g \circ f)(\mu) = (T(f) \circ T(g))(\mu)$ for all $\mu \in \Omega(L_3)$.

Let $\mu \in J(L_3)$. Then the linearity of the extension $T(g)$ and $T(g \circ f)$ to $J(L_3)$ and $T(f)$ to $J(L_2)$ and the fact that $J(L_3) = \mathrm{lin}(\Omega(L_3))$ imply that

$$T(g \circ f)(\mu) = (T(f) \circ T(g))(\mu).$$

Hence

$$T(g \circ f) = T(f) \circ T(g).$$

□

Lemma 5.10. *Every morphism in $\mathcal{A}$ is an injective set map.*

Proof. Let $f : L_1 \to L_2$ be a morphism in $\mathcal{A}$ where $L_i = (L_i, \perp_i, \oplus_i, 0_i, 1_i)$ for $i = 1, 2$.

It suffices to show that $a, b \in L_1$ and $f(a) \leq_2 f(b)$ imply $a \leq_1 b$. In fact, since

$$((f(b))')' \oplus_2 (f(b))' = 1_2 = f(b) \oplus_2 (f(b))'$$

the property (5) implies $((f(b))')' = f(b)$. But $(f(b))' = f(b')$. Then the property (2) implies that $f(a) \perp_2 f(b')$, and therefore $a \perp_1 b'$. So $a \leq_1 b$.

□

Now let T be the contravariant functor from $\mathcal{A}$ to $\mathcal{B}$ given by Theorem 5.9. The following important and main result requires a long proof:

Theorem 5.11. *The contravariant functor T is continuous.*

Proof. Let $\mathcal{D}$ be a small category and let $F : \mathcal{D} \to \mathcal{A}$ be a covariant functor. Since, by Theorem 3.7, the category $\mathcal{A}$ is cocomplete, it follows that the functor F admits a colimit (L, τ) where L is an object of $\mathcal{A}$ and $\tau = (\tau_j : F(j) \to L)_{j \in \mathrm{Ob}\,\mathcal{D}}$ is a family of morphisms in $\mathcal{A}$ satisfying:

(1) The naturality condition: If $u : j \to k$ is a morphism in $\mathcal{D}$, then $\tau_j = \tau_k \circ F(u)$.

(2) The universal property.

Since the contravariant functor $T \circ F : \mathcal{D} \to \mathcal{B}$ can be described as a covariant functor $\mathcal{D}^* \to \mathcal{B}$ and $\mathcal{B}$ is a complete category by Theorem 4.15, it follows that the functor $T \circ F$ admits a limit $((V, K), \lambda)$ where (V, K) is an object of $\mathcal{B}$ and $\lambda = (\lambda_j : (V, K) \to (T \circ F)(j))_{j \in \mathrm{Ob}\,\mathcal{D}}$ is a family of morphisms in $\mathcal{B}$ satisfying:

1'. The naturality condition: If $u : j \to k$ is a morphism in $\mathcal{D}$, then $\lambda_j = (T \circ f)(u) \circ \lambda_k$ because T is a contravariant functor.

2'. The universal property.

Since $T(L)$ is an object of $\mathcal{B}$ and $T(\tau_j) : T(L) \to (T \circ f)(j)$ are morphisms in $\mathcal{B}$ satisfying the condition (1') because of the condition (1), it follows from property (2') that there exists a unique morphism $g : T(L) \to (V, K)$ in $\mathcal{B}$ such that

$$T(\tau_j) = \lambda_j \circ g$$

for all $j \in \mathrm{Ob}\,\mathcal{D}$. By [1, Poposition 11.7] it remains to show that g is an isomorphism in $\mathcal{B}$.

Write $F(j) = L_j$ for all $j \in \mathrm{Ob}\,\mathcal{D}$. Then $(T \circ F)(j) = T(L_j)$ for all $j \in \mathrm{Ob}\,\mathcal{D}$. Moreover, by the definition of the contravariant functor T, we have $T(L) = (J(L), \Omega(L))$ and $T(L_j) = (J(L_j), \Omega(L_j))$ for all $j \in \mathrm{Ob}\,\mathcal{D}$.

The remainder of the proof is divided in several claims:

(A) Let $x, y \in K$. If $\lambda_j(x) = \lambda_j(y)$ for all $j \in \mathrm{Ob}\,\mathcal{D}$, then $x = y$.

In fact, by [14, Theorem 2, p. 109] it follows that, for every $j \in \mathrm{Ob}\,\mathcal{D}$, the following diagram:

$$\begin{tikzcd}
 & T(L_j) \\
\prod_{j\in J} T(L_j) \arrow[ur, "\mathrm{pr}_j"] & (V,K) \arrow[u, "\lambda_j"'] \arrow[l, "e"]
\end{tikzcd}$$

is commutative, where e is an equalizer of two morphisms in $\mathcal{B}$. Then

$$\lambda_j = \mathrm{pr}_j \circ e$$

for all $j \in \mathrm{Ob}\,\mathcal{D}$, and therefore

$$\mathrm{pr}_j(e(x)) = \mathrm{pr}_j(e(y))$$

for all $j \in \mathrm{Ob}\,\mathcal{D}$. Then $e(x) = e(y)$ and Remark 4.14 implies $x = y$.

(B) Let $v, w \in L$ be such that $v \perp w$. Then there exist $j \in \operatorname{Ob}\mathcal{D}$ and $a, b \in L_j$ such that $a \perp_j b$, $\tau_j(a) = v$ and $\tau_j(b) = w$.

In fact, by the dual statement of [14, Theorem 2, p. 109] it follows that, for every $j \in \operatorname{Ob}\mathcal{D}$, the following diagram:

$$\begin{tikzcd} & L_j \arrow[dl, "u_j"'] \arrow[d, "\tau_j"] \\ \coprod_{j\in J} L_j \arrow[r, "h"'] & L \end{tikzcd}$$

is commutative, where h is a coequalizer of two morphisms in $\mathcal{A}$. By Remark 3.6, h is a surjective map. Then there exist two elements $\bar{x}, \bar{y}$ of $\coprod_{j\in J} L_j$ such that $h(\bar{x}) = v$ and $h(\bar{y}) = w$. Since $v \perp w$, it follows that $\bar{x}$ and $\bar{y}$ are orthogonal elements of $\coprod_{j\in J} L_j$. Then Remark 3.4 (2) implies that there exist $j \in \operatorname{Ob}\mathcal{D}$ and $a, b \in L_j$ such that $a \perp_j b$, $\bar{x} = u_j(a)$ and $\bar{y} = u_j(b)$. Since $\tau_j = h \circ u_j$, we have $\tau_j(a) = v$ and $\tau_j(b) = w$.

(C) Let $i, j, k \in \operatorname{Ob}\mathcal{D}$, let $a \in L_i$ and let $b \in L_j$ be such that $\tau_i(a) = \tau_j(b)$. If $f_{ik} = F(i \to k)$ and $f_{jk} = F(j \to k)$, then $f_{ik}(a) = f_{jk}(b)$.

In fact, since

$$\tau_i = \tau_k \circ f_{ik}$$

$$\tau_j = \tau_k \circ f_{jk}$$

we have $\tau_k(f_{ik}(a)) = \tau_k(f_{jk}(b))$ and Lemma 5.10 implies $f_{ik}(a) = f_{jk}(b)$.

(D) Let $i,\ j \in \operatorname{Ob} \mathcal{D}$, let $a \in L_i$ and let $b \in L_j$ be such that $\tau_i(a) = \tau_j(b)$. Then $\lambda_i(x)(a) = \lambda_j(x)(b)$ for all $x \in K$.

In fact, let $k \in \operatorname{Ob} \mathcal{D}$. Then

$$\lambda_i = T(f_{ik}) \circ \lambda_k$$

$$\lambda_j = T(f_{jk}) \circ \lambda_k.$$

Since $x \in K$, we have $\lambda_i(x) \in \Omega(L_i)$. Then by the definition of the functor T, it follows that

$$T(f_{ik})(\lambda_k(x)) = \lambda_k(x) \circ f_{ik}$$

and using the part (C), we have

$$\begin{aligned}\lambda_i(x)(a) &= (T(f_{ik})(\lambda_k(x)))(a)\\ &= \lambda_k(x)(f_{ik}(a))\\ &= \lambda_k(x)(f_{jk}(b))\\ &= (T(f_{jk})(\lambda_k(x)))(b)\\ &= \lambda_j(x)(b).\end{aligned}$$

(E) Let $x \in K$. Define $l(x) : L \to \mathbb{R}$ as follows: Let $v \in L$. Since $0 \perp v$, the part (B) implies that there exist $i \in \operatorname{Ob} \mathcal{D}$ and $a \in L_i$ such that $\tau_i(a) = v$. Put

$$l(x)(v) = \lambda_i(x)(a).$$

By part (D) the map $l(x)$ is well-defined.

(F) $l(x) \in \Omega(L)$ for all $x \in K$.

In fact, we have

(i) Let $v,\ w \in L$ be such that $v \perp w$. By part (B) there exist $i \in \operatorname{Ob} \mathcal{D}$

and $a, b \in L_i$ such that $a \perp_i b$, $\tau_i(a) = v$ and $\tau_i(b) = w$. Then $v \oplus w = \tau_i(a \oplus_i b)$ and therefore

$$\begin{aligned} l(x)(v \oplus w) &= \lambda_i(x)(a \oplus_i b) \\ &= \lambda_i(x)(a) + \lambda_i(x)(b) \\ &= l(x)(v) + l(x)(w). \end{aligned}$$

(ii) $l(x)(1) = \lambda_i(x)(1_i) = 1$.

(iii) Let $v \in L$. Then there exist $i \in \operatorname{Ob}\mathcal{D}$ and $a \in L_i$ such that $\tau_i(a) = v$. Then

$$l(x)(v) = \lambda_i(x)(a)$$

and since $\lambda_i(x) \in \Omega(L_i)$ we have

$$l(x)(v) \geq 0 \text{ for all } v \in L.$$

(G) The map $l : K \to \Omega(L)$ is affine.

In fact, it follows from the fact that the map $\lambda_i|_K \colon K \to \Omega(L_i)$ is affine for all $i \in \operatorname{Ob}\mathcal{D}$.

By Lemma 4.10, there exists a unique linear extension of l to a map from V to $J(L)$ that we denote by the same symbol, and this extension is a morphism in $\mathcal{B}$.

(H) $g \circ l = 1_V$.

It suffices to verify this for the elements of K.

Let $x \in K$ and let $i \in \operatorname{Ob}\mathcal{D}$. Since

$$\lambda_i \circ g = T(\tau_i)$$

we have

$$\lambda_i(g(l(x))) = T(\tau_i)l(x).$$

So, since $l(x) \in \Omega(L)$, we get $T(\tau_i)l(x) \in \Omega(L_i)$. Let $a \in L_i$. Then

$$\begin{aligned}(T(\tau_i)l(x))(a) &= l(x)(\tau_i(a)) \\ &= \lambda_i(x)(a) \text{ for all } a \in L_i.\end{aligned}$$

So

$$T(\tau_i)l(x) = \lambda_i(x)$$

and therefore

$$\lambda_i(g(l(x))) = \lambda_i(x) \text{ for all } i \in \operatorname{Ob} \mathcal{D}.$$

Then part (A) implies $g(l(x)) = x$.

(I) $l \circ g = 1_{J(L)}$.

If suffices to verify this for the elements of $\Omega(L)$.

Let $\mu \in \Omega(L)$ and let $v \in L$. Then there exist $i \in \operatorname{Ob} \mathcal{D}$ and $a \in L_i$ such that $v = \tau_i(a)$. Hence

$$l(g(\mu))(v) = \lambda_i(g(\mu))(a)$$

and since $\lambda_i \circ g = T(\tau_i)$, we have

$$\begin{aligned}l(g(\mu))(v) &= (T(\tau_i)(\mu))(a) \\ &= (\mu \circ \tau_i)(a) \\ &= \mu(v)\end{aligned}$$

for all $v \in L$. Then $l(g(\mu)) = \mu$. □

Corollary 5.12. *Let I be a directed set and let $(F(i), f_{ij})_I$ be an inductive system in $\mathcal{A}$ over I. Then*

$$T\left(\lim_{\substack{\rightarrow \\ i \in I}} F(i)\right) \cong \lim_{\substack{\leftarrow \\ i \in I}} (T \circ F)(i).$$

Proof. Define $g_{ij} = T(f_{ij})$ for all $i, j \in I$ with $i \leq j$. Then it suffices to verify that $((T \circ F)(i), g_{ij})_I$ is a projective system in $\mathcal{B}$ over I and to apply Theorem 5.11. $\square$

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